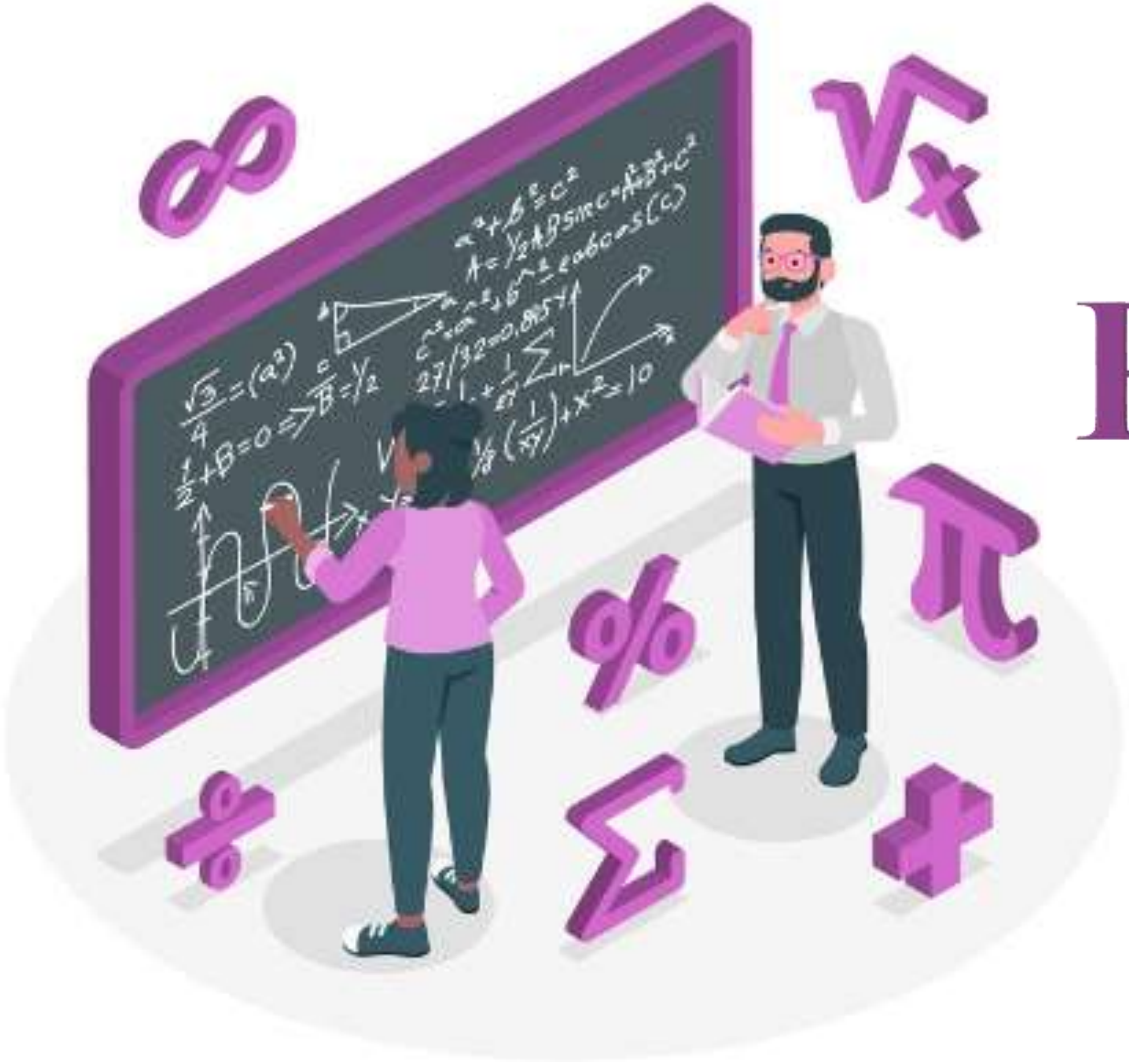

Polynomial

Prerequisites

Consider a polynomial P(x).

- Degree of P(x) is equal to the highest exponent of the variable x in P(x).
- P(x) is divisible by Q(x) if and only if exist a polynomial R(x) such that: P(x) = R(x) × Q(x)
- α is a root of P(x) (solution or zero):
  - P(α) = 0.
  - P(x) is divisible by $x - \alpha$, then P(x) can be written in form of P(x) = (x - α)Q(x) where deg(Q) = deg(P) - 1
  - Remainder of the Euclidean division of P(x) by Q(x) is zero.

Euclidean Division (Long division)

Euclidean division is a method to divide a polynomial $P(x)$ by another Polynomial $Q(x)$ where $deg(P) \geq deg(Q)$

How to perform Long Division?

Euclidean Division (Long division)

Before, we know that if we divide a number by another using long division, the Euclidean formula is: Dividend = divisor × quotient + remainder
And this formula is unique.
This process is same for the polynomials but there is a little difference.

Euclidean Division (Long division)

Example 1: Consider the two polynomials $P(x) = x^3 + 2x^2 + 3x - 1$ and $Q(x) = x - 1$.

$x^3 + 2x^2 + 3x - 1$	$x - 1$

The two polynomials must be written in decreasing order of exponent of the variable x

Euclidean Division (Long division)

Example 1: Consider the two polynomials $P(x) = x^3 + 2x^2 + 3x - 1$ and $Q(x) = x - 1$.

$$\begin{array}{r|l} x^3 + 2x^2 + 3x - 1 & x - 1 \\ \hline - \quad x^3 - x^2 & x^2 \\ \hline 3x^2 + 3x - 1 & \end{array}$$

$$\frac{x^3}{x} = x^2$$

Euclidean Division (Long division)

Example 1: Consider the two polynomials $P(x) = x^3 + 2x^2 + 3x - 1$ and $Q(x) = x - 1$.

$$\begin{array}{r|l} x^3 + 2x^2 + 3x - 1 & x - 1 \\ \hline - \quad x^3 - x^2 & x^2 + 3x \\ \hline 3x^2 + 3x - 1 & \\ - \quad 3x^2 - 3x & \\ \hline 6x - 1 & \end{array}$$

$$\frac{3x^2}{x} = 3x$$

Euclidean Division (Long division)

Example 1: Consider the two polynomials $P(x) = x^3 + 2x^2 + 3x - 1$ and $Q(x) = x - 1$.

$$
\begin{array}{r|l}
x^3 + 2x^2 + 3x - 1 & x - 1 \\
\hline
- \quad x^3 - x^2 & x^2 + 3x + 6 \\
\hline
3x^2 + 3x - 1 & \\
- \quad 3x^2 - 3x & \\
\hline
6x - 1 & \\
- \quad 6x - 6 & \\
\hline
5 &
\end{array}
$$

$$\frac{6x}{x} = 6$$

Euclidean Division (Long division)

Example 1: Consider the two polynomials $P(x) = x^3 + 2x^2 + 3x - 1$ and $Q(x) = x - 1$.

$$
\begin{array}{r|l}
x^3 + 2x^2 + 3x - 1 & x - 1 \\
\hline
-\ (x^3 - x^2) & x^2 + 3x + 6 \\
\hline
3x^2 + 3x - 1 & \\
-\ (3x^2 - 3x) & \\
\hline
6x - 1 & \\
-\ (6x - 6) & \\
\hline
5 &
\end{array}
$$

Dividend: P(x)
Divisor: Q(x)
Quotient: $x^2 + 3x + 6$
Remainder: 5

$$P(x) = (x^2 + 3x + 6)(x - 1) + 5$$

Euclidean Division (Long division)

Example 1: Consider the two polynomials $P(x) = x^3 + 2x^2 + 3x - 1$ and $Q(x) = x - 1$.

$$
\begin{array}{r|l}
x^3 + 2x^2 + 3x - 1 & x - 1 \\
\hline
-\quad x^3 - x^2 & x^2 + 3x + 6 \\
\hline
3x^2 + 3x - 1 & \\
-\quad 3x^2 - 3x & \\
\hline
6x - 1 & \\
-\quad 6x - 6 & \\
\hline
5 &
\end{array}
$$

Remark:

Remainder is $5 \neq 0$ so:

- P(x) is not divisible by Q(x)
- 1 is not a root of P(x).

Euclidean Division (Long division)

Example 2:
Consider the two polynomials:
$P(x) = -2x^4 + 5x^3 + 5x^2 - 5x - 3$
and
$Q(x) = 2x + 1.$

$$
\begin{array}{r|l}
-2x^4 + 5x^3 + 5x^2 - 5x - 3 & 2x + 1 \\ \hline
-\;\; -2x^4 - x^3 \qquad\qquad\qquad\qquad & -x^3 \\ \hline
6x^3 + 5x^2 - 5x - 3 \qquad &
\end{array}
$$

Euclidean Division (Long division)

Example 2:
Consider the two polynomials:
$P(x) = -2x^4 + 5x^3 + 5x^2 - 5x - 3$
and
$Q(x) = 2x + 1.$

$$
\begin{array}{r|l}
-2x^4 + 5x^3 + 5x^2 - 5x - 3 & 2x + 1 \\ \hline
-\ \ -2x^4 - x^3 \qquad\qquad\qquad\qquad & -x^3 + 3x^2 \\ \hline
6x^3 + 5x^2 - 5x - 3 & \\
-\ \ 6x^3 + 3x^2 \qquad\qquad & \\ \hline
2x^2 - 5x - 3 &
\end{array}
$$

Euclidean Division (Long division)

Example 2:
Consider the two polynomials:
$P(x) = -2x^4 + 5x^3 + 5x^2 - 5x - 3$
and
$Q(x) = 2x + 1.$

$$
\begin{array}{r|l}
-2x^4 + 5x^3 + 5x^2 - 5x - 3 & 2x + 1 \\
\hline
-\;\; -2x^4 - x^3 & -x^3 + 3x^2 \\
\hline
6x^3 + 5x^2 - 5x - 3 & +x \\
-\;\; 6x^3 + 3x^2 & \\
\hline
2x^2 - 5x - 3 & \\
-\;\; 2x^2 + x & \\
\hline
-6x - 3 &
\end{array}
$$

Euclidean Division (Long division)

Example 2:
Consider the two polynomials:
$P(x) = -2x^4 + 5x^3 + 5x^2 - 5x - 3$
and
$Q(x) = 2x + 1.$

$$
\begin{array}{r|l}
-2x^4 + 5x^3 + 5x^2 - 5x - 3 & 2x + 1 \\ \hline
-\quad -2x^4 - x^3 \qquad\qquad\qquad & -x^3 + 3x^2 \\
\hline
6x^3 + 5x^2 - 5x - 3 & +x - 3 \\
-\quad 6x^3 + 3x^2 \qquad\qquad & \\
\hline
2x^2 - 5x - 3 & \\
-\quad 2x^2 + x \qquad & \\
\hline
-6x - 3 & \\
-\quad -6x - 3 & \\
\hline
0 &
\end{array}
$$

Remark:
Remainder is 0 so:

- P(x) is divisible by Q(x).
- $-\frac{1}{2}$ is a root of P(x).
- $P(x) = (2x + 1)Q(x)$

Rational function

- It is in form of $f(x) = \frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$
- Euclidean division can help to simplify f(x) or to write it in another form.

Example ❶:

$$f(x) = \frac{x^3-3x^2+x+1}{x^2-1}$$

Notice that 1 is a root of the numerator: $(1)^3 - 3(1)^2 + 1 + 1 = 0$

So it is divisible by x – 1

Using Euclidean division we get:

$$x^3 - 3x^2 + x + 1 = (x-1)(x^2 - 2x - 1)$$

$$f(x) = \frac{(x-1)(x^2-2x-1)}{(x-1)(x+1)} = \frac{x^2-2x-1}{x+1} \quad ; x \neq \pm 1$$

Rational function

- It is in form of $f(x) = \frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$
- Euclidean division can help to simplify f(x) or to write it in another form.

Example ❷:

$$f(x) = \frac{x^2-2x-1}{x+1}$$

We can change the form of f(x):

$$f(x) = \frac{(x+1)(x-3)+2}{x+1} =$$

$$= \frac{(x+1)(x-3)}{x+1} + \frac{2}{x+1}$$

$$= x - 3 + \frac{2}{x+1} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

$$\begin{array}{r|l} x^2 - 2x - 1 & x + 1 \\ \hline -\quad x^2 + x & x - 3 \\ \hline -\quad -3x - 1 & \\ -3x - 3 & \\ \hline 2 & \end{array}$$

Application

Consider the polynomial $P(x) = x^3 + mx^2 + 7x + 3$
1) Find m so that $x + 3$ is a factor of $P(x)$.
2) Factorize $P(x)$.

1) $x + 3$ is a factor of $P(x)$ so -3 is a root.

$P(-3) = 0$

$(-3)^3 + m(-3)^2 + 7(-3) + 3 = 0$

$-27 + 9m - 21 + 3 = 0$

$9m - 45 = 0$

$m = \frac{45}{9} = 5$

Application

Consider the polynomial $P(x) = x^3 + mx^2 + 7x + 3$

1) Find m so that $x + 3$ is a factor of $P(x)$.

2) Factorize $P(x)$.

2) Dividing $P(x)$ by $x + 3$ using long division we get:

$P(x) = (x + 3)(x^2 + 2x + 1) = (x + 3)(x + 1)^2$